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XLVI. *Calculations in Spherical Trigonometry abridged.*
 By Israel Lyons. *In a Letter to Sir John Pringle,*
Bart. P. R. S.

TO SIR JOHN PRINGLE, BART. P. R. S.

SIR,

Redde, July 6,
 1775.

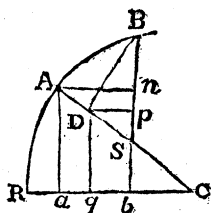
SINCE astronomical observations have been made with much greater precision than formerly, it became requisite that the calculations corresponding to them should likewise be made to much greater degrees of exactness. The ancient astronomers desired only to make their observations and computations agree within a part of a degree; succeeding ones were satisfied when they corresponded within a minute; but no less exactness than seconds will content the moderns. The rules in spherical trigonometry being reduced to operations by logarithms, it is necessary to use such a number of figures in the tables as will produce the required precision; this is very different in the various parts of the quadrant, insomuch that if the arc is only one degree, four places of decimals in the logarithm of a sine are sufficient to determine the arc to which it belongs within a second: whereas if the arc is 89° , there is a necessity

necessity of using eight figures for the same purpose: thus, the logarithm sine of $89^{\circ} 0' 0''$ is 9.9999338, the same seven figures as for the logarithm sine of $89^{\circ} 0' 1''$. From this consideration it follows, that the analogies commonly laid down and used for the solutions of spherical triangles are not in all cases equally convenient, and I might say, equally accurate; and that it would be more easy and exact in calculations to find what was required, by means of sines of arcs, which, being small, require the use of only a few places of figures. Now the cases which often occur in astronomy, where spherical trigonometry can only be of use, are generally of such a nature that we know nearly, or at least within a few degrees, what the required side or angle is, there is nothing therefore wanted but to find how much this quantity, or first approximation, differs from the true value of the side or angle. Thus in calculating the right ascension of any point of the ecliptic, whose longitude and declination are known, instead of finding the right ascension immediately, it will be more convenient to seek for the difference between the longitude and right ascension, which as it never exceeds $2\frac{1}{2}^{\circ}$, four or five places of figures will always be sufficient to determine it within a second. And in other similar cases, rules might be made agreeable to the exigency of each particular case, which would be better than the application of the general method of solution. Some examples of which shall be shewn in the

following paper: the design of which is to point out a method of solving several of the most useful questions in spherical trigonometry in a manner somewhat similar to that used in approximating to the roots of algebraic equations. This method is founded on the following

L E M M A.

If the radius is supposed equal to unity, the sine of the sum of two arcs, α and β , is equal to $\sin. \alpha + \cos. \alpha \times \sin. \beta - \sin. \alpha \times \text{vers.} \sin. \beta$. And its cosine = $\cos. \alpha - \sin. \alpha \times \sin. \beta - \cos. \alpha \times \text{vers.} \sin. \beta$.



D E M O N S T R A T I O N.

Let the arc α be RA, and the arc β be AB, their sines Aa, Bd, respectively; then Bb being drawn perpendicular to the radius CR will be the sine of $\alpha + \beta$. Draw Dp and An parallel to CR. Then, by similar triangles, CA : Ca :: BD : Bp, and CA : Aa :: AD : np. Therefore, $Bb (= Aa + Bp - pn) = Aa + \frac{Ca \times BD}{CA} - \frac{Aa \times AD}{CA}$; that is, $\overline{\sin. \alpha + \beta} = \sin. \alpha + \cos. \alpha \times \sin. \beta - \sin. \alpha \times \text{vers.} \sin. \beta$.

In the same manner, drawing Dq parallel to Aa we may prove $cb (= Ca - bq - aq) = Ca - \frac{Aa \times BD}{CA} - \frac{Ca \times AD}{CA}$, or $\overline{\cos. \alpha + \beta} = \cos. \alpha - \sin. \alpha \times \sin. \beta - \cos. \alpha \times \text{vers.} \sin. \beta$.

In

In what follows, for brevity sake, the arc is expressed by a Greek letter; its sine by the capital character; and the cosine by the small italic character of the same letter. In this notation, the two theorems will stand thus, $\sin. \overline{\alpha + \beta} = A + aB - A \times \text{vf. } \beta$, and $\cos. \overline{\alpha + \beta} = a - AB - a \times \text{vf. } \beta$.

C O R O L L A R Y I,

Since the tangent is equal to the sine divided by the cosine, we shall have

$$\text{Tang. } \overline{\alpha + \beta} = \frac{A + aB - A \times \text{vf. } \beta}{a - AB - a \times \text{vf. } \beta} = \frac{A}{a} + \frac{B}{a^2} + \frac{A}{a^3} \times \text{vf. } \beta \text{ nearly.}$$

C O R O L L A R Y II.

If we change the sign of β , we shall have $\sin. \alpha - \beta = A - aB - A \times \text{vf. } \beta$. $\cos. \alpha - \beta = a + AB - a \times \text{vf. } \beta$. And $\text{tang. } \alpha - \beta = \frac{A}{a} - \frac{B}{a^2} + \frac{A}{a^3} \times \text{vf. } \beta$.

By the help of these theorems, knowing nearly what any quantity in a spherical triangle is, we may find its correction, thus: if we have to find the cosine of an arc, which arc we know is nearly equal to α whose cosine is a . Suppose the arc to be $\alpha - \beta$, and its cosine $a + c$. Then $a + c = \cos. \alpha - \beta = a + AB - a \times \text{vf. } \beta$. Therefore, $B = \frac{c}{A} + \frac{a}{A} \times \text{verf. } \beta$.

The first term $\frac{c}{A}$ will always give a near approximation to the value of $\sin. \beta$, and β being found the correction, $\frac{a}{A} \times \text{vf. } \beta$, or $\cot. \alpha \times \text{vf. } \beta$, may be found and added to it.

Among the tables requisite to be used with the Nautical Almanac, is table iv. for parallax, p. 19. which shews the value at sight of such quantities as $\text{vf. } \beta \times \cot. \alpha$, the arc β being found in the first column of the table, and α at the top. This table I have calculated only to arcs under $63'$; but it would be found useful to have a table ready computed for all arcs under 5° .

P R O B L E M I.

If the two legs, AB and BC, of the spherical triangle ABC right-angled at B, are given, to find the hypotenuse AC, the leg BC, being A small in comparison of AC.



Let $AB = \alpha$, $BC = \beta$, and suppose $AC = \alpha + \zeta$, α being a near approximation to AC, and ζ the small arc to be added to AB to make it equal to AC; then $\cos. AC = \cos. AB \times \cos. BC$; that is, according to our notation, $a - AZ - \frac{a \times \text{vf. } \zeta}{A} = a \times \text{vf. } \zeta$.

Whence $z = \frac{a - ab}{A} - \frac{a}{A} \times \text{vf. } \zeta = \cot. \alpha \times \text{vf. } \beta - \cot. \alpha \times \text{vf. } \zeta$.

E X A M P L E.

Let AB be $75^\circ 0'$ and BC $20^\circ 0'$, and the computation will be as follows:

Cotangent AB	9.4280
Verfed sine BC	8.7804

ζ nearly $55' 33''$ fine 8.2084

Correction -7 from tab. iv. Nautical Almanac.

Therefore $\zeta = 55' 26''$ and $AC = 75^\circ 55' 26''$.

By this problem, the distance of the Sun may be found from a planet whose latitude and difference of longitude are known.

P R O B L E M II.

Having the hypotenuse AC and one of the angles A, to find the base AB.

Let $AC = \beta$, $BAC = \alpha$, and suppose $AB = \beta - \zeta$, then $\text{cof. } A = \cot. AC \times \text{tang. } AB$, or $a = \frac{b}{B} \times \frac{B}{b} - \frac{z}{b^2} + \frac{B \times \text{vf. } \zeta}{b^3} = 1 - \frac{z}{Bb} + \frac{B \times \text{vf. } \zeta}{b^2}$.

Whence $z = Bb \times 1 - a + \frac{B}{b} \times \text{vf. } \zeta = \frac{1}{2} \text{ fin. } 2 \beta \times \text{vf. } \alpha + \text{tang. } \beta \times \text{vf. } \zeta$.

E X A M P L E.

Let $A = 23^\circ 28' 15''$, and $AC = 10^\circ 0' 0''$.

Sine 2 AC $20^\circ 0'$ 9.5340

Verfed fine A 8.9177

8.4517

Log. 2. 0.3010

ζ nearly 48' 39" fine 8.1507

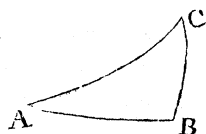
Correction + 6

ζ 48 45 and $BC = 159^\circ 11' 15''$.

By this problem, the right ascension of any point of the ecliptic, whose obliquity and longitude are known, may be found.

P R O B L E M III.

Supposing the same things known as in the last, to find the perpendicular BC, when the hypotenuse is nearly a quadrant.



Let $A = \alpha$, $AC = \beta$, as before, and suppose $BC = a - \zeta$; then
 $\sin. BC = \sin. AC \times \sin. A$, or $A - a Z - A \times \text{vf. } \zeta = AB$, whence
 $z = \frac{A - BA}{a} - \frac{A}{a} \times \text{vf. } \zeta = \text{tang. } \alpha \times \text{co. ver. fin. } \beta - t. \alpha \times \text{vf. } \zeta$.

E X A M P L E.

Let $A = 23^\circ 28' 15''$, and $AC = 80^\circ 0'$.

Tang. A	9.6377
Verf. fin. co. AC, 10°	8.1816

ζ nearly	$22^\circ 41'$ fine	7.8193
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Correction	- 1
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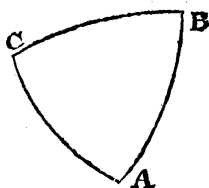
ζ $22^\circ 40'$ and $BC = 23^\circ 5' 35''$.

This problem will be of use to find the declination of the ecliptic, and the latitude of a planet near the limits.

These three instances will suffice for an application of this method to right-angled spherical triangles; we shall now give two problems of oblique triangles.

PROBLEM IV.

Suppose ABC to be a spherical triangle, in which are given the two sides AB, BC, with the included angle B, to find the third side AC.



SOLUTION I.

Let $ABC = \beta$, $BC = \alpha$, $AB = \delta$. Put $AC = \beta + \zeta$, β being an approximate value of AC, when the two legs are nearly quadrants. Now the cosine of AC being equal to $bDA + da^{(a)}$ we shall have $b - BZ - b \times \text{vf. } \zeta = bDA + da$: and $z = \frac{b - bDA - da}{B} - \frac{b}{B} \times \text{vf. } \zeta$. But $1 - DA - da = \text{vf. } \delta - \alpha$, which put = w . Then $z = \frac{bw}{B} + \frac{bda - da}{B} - \frac{b}{B} \times \text{vf. } \zeta = \cot. \beta \times \text{vf. } \delta - \alpha - \cot. \delta \times \cot. \alpha \times \text{tang. } \frac{1}{2}\beta - \cot. \beta \times \text{vf. } \zeta$. Therefore ζ is the difference of two arcs whose sines are $\cot. \beta \times \text{vf. } \delta - \alpha$, and $\cot. \delta \times \cot. \alpha \times \text{tang. } \frac{1}{2}\beta$, the difference of these two arcs being diminished by the correction $\cot. \beta \times \text{vf. } \zeta$.

(a) It is a well known theor. that $\sin. BA \times \sin. BC : r^2 = \text{vf. } AC - \text{vf. } AB - BC : \text{vf. } B$; that is, $\sin. BA \times \sin. BC : r^2 = \cot. AB - BC - \cot. AC : r - \cot. B$. Or, in the author's notation, putting $r = 1$, $DA : 1 = \cot. \delta - \alpha - \cot. AC : 1 - b$. Therefore $DA - bDA = \cot. \delta - \alpha - \cot. AC$. Or, $\cot. AC = bDA - DA - \cot. \delta - \alpha$. For $\cot. \delta - \alpha$ substitute its value as expressed in the second corollary of the lemma, and there arises the author's equation, $\cot. AC = bDA + da$.

S. HORSLEY.

EXAMPLE.

E X A M P L E.

Suppose $B = 51^{\circ} 12' 5''$

$AB = 87 \quad 57 \quad 51$

$BC = 87 \quad 20 \quad 34$

Cotangent B 9.9053

Verf. fine $AB - BC \quad 0^{\circ} 37' 17'' \quad 5.7693$

Tang. $\frac{1}{2} B \quad 25^{\circ} 36' \quad 9.6804$

Cofine AB 8.5506

Cofine BC 8.6661

1st arc $0^{\circ} 10''$ fine 5.6746

2d arc $2^{\circ} 43''$ fine 6.8971

The difference of these two arcs,

Subtracted from the value of the angle B, $51 \quad 12 \quad 5$

Leaves AC,

$51 \quad 9 \quad 32$

The correction $\cot. \beta \times \text{vf. } \zeta$ in this example is 0.

This solution is very convenient to find the distance of two Zodiacal Stars, having their latitudes and difference of longitude.

S O L U T I O N II.

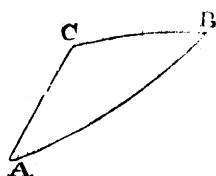
Let τ be an arc whose cofine $t = b \times \text{cof. } \delta - \alpha = b d a + b d A$, and suppose $AC = \tau - \zeta$, then $t + \tau Z - t \times \text{vf. } \zeta = b d A + d a = t - b d a + d a$. Whence $z = d a \times \frac{1-b}{\tau} + \frac{t}{\tau} \times \text{vf. } \zeta = \text{cofec. } \tau \times \text{cofin. } \alpha \times \text{cofin. } \delta \times \text{vf. } \beta + \cot. \tau \times \text{vf. } \zeta$.

This solution is useful to find the distance of the Moon from a star at some distance from the ecliptic, in which case it coincides with the rule given by the Astronomer

Royal, Phil. Transf. 1764, vol. LIV. and which taking in the correction here given $\cot. \tau \times \text{vf. } \zeta$ will always be exact to a second. It is also of use to find the declination of a star, whose longitude and latitude and obliquity of the ecliptic are known.

S O L U T I O N III.

Let the angle B be small, and the two legs AB, BC, very unequal; then the side AC will be nearly $AB - BC$. Put this $= k$, and suppose $AC = k + \zeta$, then $\cos.$



$AC = k - KZ - k \times \text{vf. } \zeta = ad + AD - KZ - k \times \text{vf. } \zeta = bAD + ad,$
 Whence $z = \frac{DA - bAD}{K} - \frac{k}{K} \times \text{vf. } \zeta = \sin. \delta \times \sin. \alpha \times \text{vf. } \beta \times \text{cosec.}$
 $\delta = \alpha - \cot. k \times \text{vf. } \zeta.$

E X A M P L E.

Let $AB = 94^\circ 36' 58''$
 $BC = 23 \quad 28 \quad 24$
 $B = 24 \quad 54 \quad 24$ } as in the example to fol. 2.

$AB - BC = 71 \quad 8 \quad 34$

Sine AB

Sine BC

Verfed of B

ζ nearly $2^\circ 14' 11''$ fine

Cosecant 0.02396

9.99859

9.60023

8.96851

8.59129

The value of ζ being without the limits of tab. IV. in the tables requisite to be used with the Nautical Almanac, the correction cot. x × vf. ζ must be computed thus:

Cot. x	9.533
V. fin. ζ	6.881
	<hr/>

Cor. $0' 53''$. sine 6.414, this subtracted from the first value of ζ , leaves $\zeta = 2^\circ 13' 18''$, which added to $\delta - \alpha$, gives the side $AC = 73^\circ 21' 52''$. This solution will help to find the Sun's altitude near noon.

I have dwelt the longer on this problem because it is one that is very commonly required in astronomical calculations, and the operation by the rules of spherical trigonometry in this as well as the next is rather troublesome.

P R O B L E M V.

Supposing the same things given, to find either of the angles, as for instance c opposite the side AB.

We have $\cot. c = \cot. B \times \cos. BC - \sin. BC \times \cot. AB \times \operatorname{cosec}. B$
 $= \frac{ba}{B} - \frac{Ad}{BD}$. Let μ be an angle whose $\cot. \frac{m}{M} = \cot. \beta \times$
 $\sin. \delta - \alpha \times \operatorname{cosec}. \delta = \frac{baD - bAd}{BD}$, and suppose $c = \mu + \zeta$, then
 $\cot c = \frac{m}{M} - \frac{z}{M^2} + \frac{m \cdot \operatorname{vf}. \zeta}{M^3} = \frac{baD - bAd}{BD}$. Whence $z = M \times \frac{Ad - bAd}{BD} +$
 $\frac{m}{M} \times \operatorname{vf}. \zeta = \sin. \mu^2 \times \sin. \alpha \times \cot. \delta \times \operatorname{tang}. \frac{1}{2} \beta + \cot. \mu \times \operatorname{vf}. \zeta$.

E X A M P L E.

Let $AB = 94^\circ 36' 58$
 $BC = 23 \ 28 \ 54$
 $B = 24 \ 54 \ 24$ Cotang. 0.3331770

Diff. AB and BC = $71 \ 8 \ 34$ Sine 9.9760412
Cofecant AB 0.0014080

 $\mu = 26 \ 3 \ 44$ Cot. 10.3166262

Sin. μ 9.286
Sin. BB 9.600
Cot. AB 8.909
Tang. $\frac{1}{2} B$ 9.344

 $\zeta = 4' 44''$ sine 7.139 , this subtracted from μ leaves the
angle $c = 25^\circ 59' 0''$

This

This problem will be of use to find the right ascension of a star whose longitude and latitude, and obliquity of the ecliptic are known, or to find the Sun's azimuth at any hour in a given latitude.

I have added no cautions when these approximations and corrections change their signs, because any mathematician will discover them at sight.

I have the honour to be, &c.